WHITTAKER FUNCTIONS ON QUANTUM GROUPS AND Q-DEFORMED TODA OPERATORS

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Dedicated to Dmitry Borisovich Fuchs on the occasion of his sixtieth birthday

Introduction

Let G be a simply connected simple Lie group over $\mathbb{C}$. Let $N_\pm$ be the positive and the negative maximal unipotent subgroups, and H the maximal torus, corresponding to some polarization of G. Let $G_0 = N_- H N_+$ be the big Bruhat cell.

Let $\chi_\pm : N_\pm \to \mathbb{C}^*$ be holomorphic nondegenerate characters (i.e. they don't vanish on simple roots). A Whittaker function on G_0 with characters χ_+, χ_- is any holomorphic function ϕ on G_0 such that $\phi(a_- a_0 a_+) = \chi_-(a_-)\chi_+(a_+)\phi(a_0)$, where $a_\pm \in N_\pm$, $a_0 \in G_0$. Thus, a Whittaker function is completely determined by its values on the maximal torus H.

In the late 1970's it was observed (by Kazhdan and Kostant) that the restriction of the Laplace operator on G to Whittaker functions is the quantum Toda Hamiltonian. This allows one to easily prove Kostant's integrability theorem for the quantum Toda system: quantum integrals are restrictions to Whittaker functions of higher Casimirs of G.

This procedure can be generalized to the case when the group G is replaced with the corresponding affine Kac-Moody group $\hat{G}$. In this case, one should consider Whittaker functions "of critical level", i.e. functions satisfying the equation $\phi(xz) = \phi(x)z^{-h^\vee}$, where z is a central element of $\hat{G}$ and $h^\vee$ the dual Coxeter number of G. The restriction of the Laplace operator to Whittaker functions is then the quantum affine Toda Hamiltonian. As a result, one gets a proof of the integrability of the quantum affine Toda system: quantum integrals are restrictions to Whittaker functions of higher Casimirs of $\hat{G}$ at the critical level defined in [FF,GW].

The goal of this paper is to generalize this theory (both for G and $\hat{G}$) to the case of quantum groups, and give the corresponding proofs of the quantum integrability of the q-difference analogs of the quantum Toda systems for G and $\hat{G}$.

The main problem with this generalization is that the algebra $U_q(\mathfrak{n}_+)$, which is the q-analogue of N_+, does not have a homomorphism to $\mathbb{C}$ which is a deformation of χ_+. This problem can be dealt with as follows.

Consider the universal character $\eta_+ : \mathfrak{n}_+ \to \mathbb{C}^r$ (where r is the rank of G), so that individual nondegenerate characters χ_+ are obtained by the formula $\chi_+ = \xi \circ \eta_+$, where $\xi : \mathbb{C}^r \to \mathbb{C}$ is a linear map whose coordinates are nonzero. (Here we abuse notation by denoting the Lie algebra homomorphism corresponding to the group homomorphism χ_+ also by χ_+). We can regard η_+ as a homomorphism from $U(\mathfrak{n}_+)$ to the polynomial ring $\mathbb{C}[x_1, ..., x_r]$. The well-known but crucial observation is that η_+ admits a quantum deformation η_+^q, which is a map from $U_q(\mathfrak{n}_+)$ to a certain quantum polynomial algebra. This allows one to generalize the definition of a Whittaker function to the q-case, after which it is more or less straightforward to generalize the results about Toda systems.

Remark. A different (but closely related) method of dealing with the problem of absence of characters is to multiply the generators of $U_q(\mathfrak{n}_+)$ by elements of the Cartan subgroup in such a way that the algebra generated by the obtained elements has nondegenerate characters. This beautiful idea was introduced in the recent paper [S], which appeared while our paper was being prepared. This approach, as indicated in [S], can also be used to produce q-deformations of quantum Toda systems.

The paper is structured as follows. In Section 1 we recall the Kazhdan-Kostant construction for finite dimensional classical groups. In Section 2 we generalize this construction to the affine case. In Section 3 we generalize the construction to finite-dimensional quantum groups using the method described above. In Section 4 we give the generalization to affine quantum groups. In Section 5 we explain how to compute explicitly the Toda Hamiltonians for quantum and affine quantum groups, and do so for type A. In Section 6 we discuss the connection of our q-deformed Toda systems with the ordinary Toda systems and their known q-deformations. In Section 7 we discuss the relation between the Toda systems and the Calogero-Moser, Macdonald, and Ruijsenaars systems.

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1. Toda systems

We start with recalling the Kazhdan-Kostant construction.

To avoid difficulties with generalizing the results to quantum groups, we will take an algebraic approach and use formal groups rather than ordinary complex Lie groups.

Let $\mathfrak{g}$ be a simple complex Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra in $\mathfrak{g}$. We fix an invariant inner product on $\mathfrak{g}$ such that long roots have squared norm 2.

Definition. *The quantum Toda Hamiltonian corresponding to the Lie algebra $\mathfrak{g}$ is the following differential operator on $\mathfrak{h}$:*

$$M = -\frac{1}{2}\Delta + \sum_{i=1}^{r} e^{\alpha_i(h)}, \tag{1.1}$$

where Δ is the Laplacian corresponding to the invariant form, and α_i are simple positive roots of $\mathfrak{g}$.

Theorem 1.1. *([K]) The operator M defines a completely integrable quantum system. More precisely, there exist differential operators $M_1, ..., M_r$ on $\mathfrak{h}$ ($r = rank(\mathfrak{g})$) such that*

(a) $[M_i, M_j] = [M_i, M] = 0$;

(b) The symbols of M_i are Weyl group invariant elements of $S\mathfrak{h}$ which generate $(S\mathfrak{h})^W$.

The proof of Theorem 1.1 occupies the rest of Section 1.

Choose a polarization $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$.

Let $C \in U(\mathfrak{g})$ be the quadratic Casimir corresponding to the invariant form on $\mathfrak{g}$. We have

$$C = \sum_{i=1}^{r} y_i^2 + 2\sum_{\alpha>0} f_\alpha e_\alpha + 2\rho, \tag{1.2}$$

where y_i is an orthonormal basis of $\mathfrak{h}$, e_α, f_α are root elements, and ρ is the element of $\mathfrak{h}$ such that $\alpha(\rho) = 1$ for all simple roots α.

Let $A_\mathfrak{g} = U(\mathfrak{g})^*$ be the algebra of functions on the formal group G associated to $\mathfrak{g}$.

For any $X \in U(\mathfrak{g})$, let $R(X), L(X) \in \mathrm{End}_\mathbb{C}(A_\mathfrak{g})$ be the right-invariant and left-invariant differential operators on G corresponding to X. Thus we get

$$R(C) = L(C) = \sum_{i=1}^{r} L(y_i)^2 + 2\sum_{\alpha>0} L(f_\alpha)L(e_\alpha) + 2L(\rho). \tag{1.3}$$

Let $A_\mathfrak{h} = U(\mathfrak{h})^*$ be the algebra of functions on the formal group H associated to $\mathfrak{h}$. Let $\phi \to \phi|_\mathfrak{h}$ denote the map $A_\mathfrak{g} \to A_\mathfrak{h}$ dual to the embedding $U(\mathfrak{h}) \to U(\mathfrak{g})$. Let ∂_h be the derivation of the algebra $A_\mathfrak{h}$ corresponding to $h \in \mathfrak{h}$. In particular, let $\partial_{y_i} = \partial_i$.

Let $\chi_\pm : U(\mathfrak{n}_\pm) \to \mathbb{C}$ be defined by $\chi_+(e_i) = \chi_+^i, \chi_-(f_i) = \chi_-^i$ (e_i, f_i are simple root vectors).

Definition. *A Whittaker function with characters χ_+, χ_- is a function $\phi \in A_\mathfrak{g}$ such that $L(X_+)\phi = \chi_+(X_+)\phi$, $R(X_-)\phi = \chi_-(X_-)\phi$, where $X_\pm \in U(\mathfrak{n}_\pm)$.*

By Frobenius reciprocity, the space of Whittaker functions can be naturally identified with $A_\mathfrak{h}$, via $\phi \to \phi|_\mathfrak{h}$.

Let $\phi \in A_\mathfrak{g}$ be a Whittaker function with characters χ_+, χ_-.

Proposition 1.2. *One has*

$$(L(C)\phi)|_\mathfrak{h} = (\sum_{i=1}^{r} \partial_i^2 + 2\partial_\rho + 2\sum_{i=1}^{r} \chi_+^i \chi_-^i e^{\alpha_i(h)})\phi|_\mathfrak{h} \tag{1.4}$$

(where $h \in \mathfrak{h}$).

Proof. Applying (1.3) to ϕ, we get

$$(L(C)\phi)|_\mathfrak{h} = (\sum_{i=1}^{r} \partial_i^2 + 2\partial_\rho)\phi|_\mathfrak{h} + (2\sum_{i=1}^{r} \chi_+^i L(f_i)\phi)|_\mathfrak{h} \tag{1.5}$$

So it remains to compute $(L(f_i)\phi)|_{\mathfrak{h}}$. We have $L(f_i)(e^h) = Ad(e^h)^{-1}(R(f_i)(e^h)) = e^{\alpha_i(h)}R(f_i)(e^h)$. Therefore, we get (1.4). □

Let D' be the differential operator on the RHS of (1.4). It is easy to see that the operator $D = e^{(\rho,h)}D'e^{-(\rho,h)}$ has the form

$$D = \sum_{i=1}^{r} \partial_i^2 + \sum_{i=1}^{r} 2\beta_i e^{\alpha_i(h)} + (\rho,\rho), \beta_i = \chi_+^i \chi_-^i. \tag{1.6}$$

Thus, if $\beta_i = -1$ then $M := -\frac{1}{2}(D - (\rho,\rho))$ is the quantum Toda Hamiltonian (1.1).

Now let us find quantum integrals of M.

Proposition 1.3. *(i) For any element $Y \in U(\mathfrak{g})$ there exists a unique differential operator $D_Y : A_{\mathfrak{h}} \to A_{\mathfrak{h}}$ on H such that for any Whittaker function $\phi \in A_{\mathfrak{g}}$ one has $(L(Y)\phi)|_{\mathfrak{h}} = D_Y\phi|_{\mathfrak{h}}$.*

(ii) If Y and Y' are central in $U(\mathfrak{g})$ then $D_{YY'} = D_Y D_{Y'}$.

Proof. (i) The proof is a straightforward generalization of the argument in the proof of Proposition 1.2.

(ii) The statement follows from the fact that if Y is central then $L(Y)$ maps Whittaker functions to themselves. □

Now let $Y_1, Y_2, ..., Y_r$ be a system of generators of $U(\mathfrak{g})^{\mathfrak{g}}$, and $M_i = e^{(\rho,h)}D_{Y_i}e^{-(\rho,h)}$. Then M_i satisfy the conditions of Theorem 1.1, Q.E.D.

2. Affine Toda systems

In this section we generalize the results of Section 1 to the affine case.

Definition. *The affine quantum Toda Hamiltonian corresponding to the Lie algebra $\mathfrak{g}$ is the following differential operator on $\mathfrak{h}$:*

$$M^K = -\frac{1}{2}\Delta + \sum_{i=1}^{r} e^{\alpha_i(h)} + Ke^{-\theta(h)}, \tag{2.1}$$

where Δ, α_i are as above, θ is the maximal root of $\mathfrak{g}$, and K is a nonzero complex number.

Remark. Note that K is an essential parameter and cannot be removed by a simple change of variables. Also observe that $M^0 = M$, where M is as in the previous section.

The following generalization of Theorem 1.1 was proved by Olshanetsky and Perelomov [OP] for Lie algebras of type A and follows from the results of Cherednik [Ch1] (see Section 7) in the general case.

Theorem 2.1. *The operator M^K defines a completely integrable quantum system. More precisely, there exist differential operators $M_1^K, ..., M_r^K$ on $\mathfrak{h}$ ($r = rank(\mathfrak{g})$) such that*

(a) $[M_i^K, M_j^K] = [M_i^K, M^K] = 0$;

(b) The symbols of M_i^K are Weyl group invariant elements of $S\mathfrak{h}$ which generate $(S\mathfrak{h})^W$.

The proof of this theorem is a generalization of the proof of Theorem 1.1, and is given below. The main modification in the proof, compared to Section 1, is

that now instead of the Lie algebra $\mathfrak{g}$ one should consider the affine Lie algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$.

Recall [Kac] that the affine Lie algebra $\hat{\mathfrak{g}}$ has the form $L\mathfrak{g} \oplus \mathbb{C}c$, where $L\mathfrak{g}$ is the Lie algebra of Laurent polynomials of a variable t with values in $\mathfrak{g}$, and c is a central element. The commutator is defined by $[f(t), g(t)] = [fg](t) + \mathrm{Res}_{t=0}(df, g)c$.

Let $\hat{\mathfrak{n}}_+$ be the Lie subalgebra of $L\mathfrak{g}$ of elements $g(t)$ regular at 0 and such that $g(0) \in \mathfrak{n}_+$. Let $\hat{\mathfrak{n}}_-$ be the Lie subalgebra of $L\mathfrak{g}$ of elements $g(t)$ regular at infinity and such that $g(\infty) \in \mathfrak{n}_-$. Let $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c$. Thus we have $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_+ \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_-$.

Let $A_{\hat{\mathfrak{g}}} = U(\hat{\mathfrak{g}})^*$ be the algebra of functions on the formal group $\hat{G}$ associated to $\hat{\mathfrak{g}}$. For any $X \in U(\hat{\mathfrak{g}})$, let $R(X), L(X)$ be the left- and right-invariant differential operators on $\hat{G}$ corresponding to X (i.e. $R(X), L(X)$ are endomorphisms of the space $A_{\hat{\mathfrak{g}}}$).

For any complex number k, let $A_{\hat{\mathfrak{g}}}^k \subset A_{\hat{\mathfrak{g}}}$ be the space of functions satisfying the equation $L(c)f = kf$. Then it is clear that for any $Y \in U(\hat{\mathfrak{g}})$, $L(Y)$ preserves $A_{\hat{\mathfrak{g}}}^k$, and $L(c)|_{A_{\hat{\mathfrak{g}}}^k} = k \cdot Id$. Thus L descends to a map $L : U(\hat{\mathfrak{g}})/(c-k) \to \mathrm{End}_{\mathbb{C}}(A_{\hat{\mathfrak{g}}}^k)$.

The value $k = -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ (the so called critical level) is especially important for us, because of the following theorem.

Let $\hat{U}(\hat{\mathfrak{g}})$ be the completion of $U(\hat{\mathfrak{g}})$ acting in the category of $\hat{\mathfrak{g}}$-modules which are locally nilpotent under the action of simple root vectors e_i, $i = 0, ..., r$ (i.e. for any vector v there exists $N(v) > 0$ such that $e_{i_1} ... e_{i_{N(v)}} v = 0$ for any $i_1, ..., i_{N(v)}$).

Theorem 2.2. *([FF,GW]) The algebra $\hat{U}(\hat{\mathfrak{g}})/(c+h^\vee)$ contains algebraically independent central elements $Y_1, ..., Y_r$, of degree 0 with respect to the t-grading, such that:*

(a) $Y_1 = \hat{C} := C + 2\sum_{m \geq 0} \sum_{a \in B} (a \otimes t^{-m})(a \otimes t^m)$, where B is an orthonormal basis of $\mathfrak{g}$;

(b) Y_i are of the form $Y_i = Y_i^0 + Y_i^+$, where Y_i^0 is a Weyl group invariant element of $S\mathfrak{h}$, and Y_i^+ is a sum of monomials of degree $\leq deg(Y_i^0)$, which belong to $U(\hat{\mathfrak{g}})\hat{\mathfrak{n}}_+$.

Now we define Whittaker functions, in the same way as before. Let $\chi_\pm : U(\hat{\mathfrak{n}}_\pm) \to \mathbb{C}$ be defined by $\chi_+(e_i) = \chi_+^i, \chi_-(f_i) = \chi_-^i$ (e_i, f_i are simple root vectors, $i = 0, ..., r$).

Definition. *A Whittaker function with characters χ_+, χ_- is a function $\phi \in A_{\hat{\mathfrak{g}}}^{-h^\vee}$ such that $L(X_+)\phi = \chi_+(X_+)\phi$, $R(X_-)\phi = \chi_-(X_-)\phi$, where $X_\pm \in U(\hat{\mathfrak{n}}_\pm)$.*

By Frobenius reciprocity, the space of Whittaker functions can be naturally identified with $A_{\mathfrak{h}}$, via $\phi \to \phi|_{\mathfrak{h}}$.

Let $\phi \in A_{\hat{\mathfrak{g}}}^{-h^\vee}$ be a Whittaker function with characters χ_+, χ_-.

Proposition 2.3. *The series $((L(\hat{C})\phi)|_{\mathfrak{h}}$ is finite. Moreover, one has*

$$(L(\hat{C})\phi)|_{\mathfrak{h}} = (\sum_{i=1}^r \partial_i^2 + 2\partial_\rho + 2\sum_{i=0}^r \chi_+^i \chi_-^i e^{\alpha_i(h)})\phi|_{\mathfrak{h}} \tag{2.2}$$

(where $h \in \mathfrak{h}$).

(The difference between the right hand sides of (1.4) and (2.2) is that in (2.2) the second summation includes $i = 0$.)

Proof. The proof is completely analogous to the proof of Proposition 1.2. □

Let $\hat{D}'$ be the differential operator on the RHS of (2.2). It is easy to see that the operator $\hat{D} = e^{(\rho,h)}\hat{D}'e^{-(\rho,h)}$ has the form

$$\hat{D} = \sum_{i=1}^{r} \partial_i^2 + \sum_{i=0}^{r} 2\beta_i e^{\alpha_i(h)} + (\rho,\rho), \beta_i = \chi_+^i \chi_-^i. \tag{2.3}$$

Thus, if $\beta_i = -1$ for $i \geq 1$, and $\beta_0 = -K$ then $M^K = -\frac{1}{2}(\hat{D} - (\rho,\rho))$ is the affine quantum Toda Hamiltonian (2.1).

Now let us find quantum integrals of M^K.

Proposition 2.4. *(i) For any $i = 1, ..., r$ the series $(L(Y_i)\phi)|_{\mathfrak{h}}$ is finite for any Whittaker function ϕ. Moreover, there exists a unique differential operator D_{Y_i} : $A_{\mathfrak{h}} \to A_{\mathfrak{h}}$ on H such that for any Whittaker function ϕ one has $(L(Y_i)\phi)|_{\mathfrak{h}} = D_{Y_i}\phi|_{\mathfrak{h}}$.*

(ii) $[D_{Y_i}, D_{Y_j}] = 0$.

Proof. The finiteness of the series follows from the fact that terms containing non-simple roots vanish. The rest of the proof is analogous to the proof of Proposition 1.3. □

Now let $M_i^K = e^{(\rho,h)}D_{Y_i}e^{-(\rho,h)}$. Then M_i^K satisfy the conditions of Theorem 2.1, Q.E.D.

3. q-deformed Toda systems

In this section we will generalize the constructions of Section 1 to the case when the classical group G is replaced with the corresponding quantum group.

Recall that H denotes the (formal) maximal torus in G, and $A_{\mathfrak{h}}$ is the algebra of regular functions on H.

Definition. *A difference operator on H is an operator on the space $A_{\mathfrak{h}}[[\hbar]]$ (where $\hbar$ is a formal parameter) which has the form $\sum f_i T_{\beta_i}$ (a finite sum), where $f_i \in A_{\mathfrak{h}}[[\hbar]]$, and $T_\beta f(x) = f(xe^{\hbar\beta})$, $\beta \in \mathfrak{h}$.*

Difference operators form an algebra, which we'll call $D_q(H)$. For any algebra B, by a B-valued difference operator we will mean an element of $D_q(H) \otimes B$ (the algebraic, i.e. uncompleted tensor product).

Below, using the method of Section 1, we will produce a commuting family of r algebraically independent scalar valued difference operators on H, which is a deformation of the Toda system.

Let $U_\hbar(\mathfrak{g})$ be the Drinfeld-Jimbo quantum universal enveloping algebra corresponding to the Lie algebra $\mathfrak{g}$ (with the formal quantization parameter). It has generators e_i, f_i, h_i and the standard relations [CP]. Let $U_\hbar(\mathfrak{n}_+)$ be the subalgebra of $U_\hbar(\mathfrak{g})$ generated by e_i, and $U_\hbar(\mathfrak{n}_-)$ be the subalgebra generated by f_i, respectively.

Let $A_{U_\hbar(\mathfrak{g})} = U_\hbar(\mathfrak{g})^*$ be the space of linear functions on $U_\hbar(\mathfrak{g})$, i.e. the dual quantum formal group. The space $A_{U_\hbar(\mathfrak{g})}$ is the quantum analog of $A_{\mathfrak{g}}$.

The obvious two-sided action of $U_\hbar(\mathfrak{g})$ on itself induces a two-sided action of $U_\hbar(\mathfrak{g})$ on $A_{U_\hbar(\mathfrak{g})}$, via $((a,b)\circ\phi)(x) = \phi(axb)$ (here the action of a is a right action and the action of b is a left action). Define the maps $L, R : U_\hbar(\mathfrak{g}) \to \mathrm{End}(A_{U_\hbar(\mathfrak{g})})$ by $(a,b)\circ\phi = R(a)L(b)\phi$.

Recall [D,R] that the center of $U_\hbar(\mathfrak{g})$ is spanned by elements C_V corresponding to finite dimensional representations V of $U_\hbar(\mathfrak{g})$ by the formula

$$C_V = \mathrm{Tr}|_V (1 \otimes \pi_V)(\mathcal{R}^{21}\mathcal{R}(1 \otimes e^{2\hbar\rho})), \tag{3.1}$$

where $\mathcal{R}$ is the universal R-matrix of $U_\hbar(\mathfrak{g})$. The map $V \to C_V$ defines a homomorphism of the Grothendieck ring of the category of finite dimensional representations of $U_\hbar(\mathfrak{g})$ and the center of $U_\hbar(\mathfrak{g})$.

Now we would like to define the notion of a Whittaker function. In order to do so, we will define an analogue of the notion of a nondegenerate character.

Let (a_{ij}) be the Cartan matrix of $\mathfrak{g}$. Let $d_i = 1, 2, 3$ be the set of relatively prime integers such that the matrix $b_{ij} = d_i a_{ij}$ is symmetric. Let ω be any orientation of the Dynkin diagram of $\mathfrak{g}$. Define the quantum polynomial algebra P_ω generated by variables $x_1, ..., x_r$ corresponding to vertices of the Dynkin diagram, with relations $x_i x_j = e^{\pm\hbar b_{ij}} x_j x_i$, where the sign is $+$ if the edge ij is oriented from i to j and $-$ otherwise.

We will need the following well known proposition.

Proposition 3.1. *There exists a homomorphism $\chi_+ : U_\hbar(\mathfrak{n}_+) \to P_\omega$ such that $\chi_+(e_i) = x_i$.*

Proof. It is sufficient to check that χ_+ respects the Serre relations. Thus, it is enough to show that

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{d_i}} x_i^{1-a_{ij}-k} x_j x_i^k = 0, \tag{3.2}$$

(here $q = e^\hbar$, and $[a]_p = \frac{p^a - p^{-a}}{p - p^{-1}}$). Using the relations between x_i, we can reduce this to

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{d_i}} q^{\pm k b_{ij}} = 0,$$

which holds by the q-binomial theorem.

We also let $\chi_- : U_\hbar(\mathfrak{n}_-) \to P_\omega$ be the homomorphism defined by $\chi_-(f_i) = x_i$.

Introduce the algebra $A_\omega = P_\omega \otimes A_{U_\hbar(\mathfrak{g})}$.

Definition. *A Whittaker function on $U_\hbar(\mathfrak{g})$ is an element $\phi \in A_\omega$ such that for any $a_\pm \in U_\hbar(\mathfrak{n}_\pm)$ one has*

$$(a_-, a_+) \circ \phi = \chi_+(a_+)\phi\chi_-(a_-). \tag{3.3}$$

It is obvious that the module $U_\hbar(\mathfrak{g})$ over $U_\hbar(\mathfrak{n}_-) \otimes U_\hbar(\mathfrak{n}_+)$ (where the first factor acts by left shifts and the second by right shifts) is freely generated by the Cartan subalgebra $U_\hbar(\mathfrak{h})$. Therefore, the space of Whittaker functions is naturally identified with $P_\omega \otimes U_\hbar(\mathfrak{h})$. As before, we will denote this identification by $\phi \to \phi|_\mathfrak{h}$.

Let $Y \in U_\hbar(\mathfrak{g})$. Then Y defines an endomorphism $L(Y)$ of $A_{U_\hbar(\mathfrak{g})}$ and A_ω. As before, we have

Proposition 3.2. *(i) For any $Y \in U_\hbar(\mathfrak{g})$, which is a noncommutative polynomial of elements $e^{\hbar x}$ $(x \in \mathfrak{h})$, e_i, and f_i, there exists a unique difference operator D_Y on H with coefficients in $B = P_\omega \otimes P_\omega^{op}$ such that $(L(Y)\phi)|_\mathfrak{h} = D_Y\phi|_\mathfrak{h}$ for any Whittaker function ϕ.*

(ii) If Y and Y' are central then $D_{YY'} = D_Y D_{Y'}$.

Proof. Same as for Proposition 1.3.

Now consider the fundamental representations $V_1, ..., V_r$ of $U_\hbar(\mathfrak{g})$ and define $Y_i = C_{V_i}$. Then D_{Y_i} are B-valued commuting difference operators

Unfortunately, operators D_{Y_i} are with coefficients in $B = P_\omega \otimes P_\omega^{op}$, while we want to obtain operators with scalar coefficients. However, since C_V has zero weight for any V, the element D_Y for $Y = C_V$ is in fact a difference operator with coefficients in the subalgebra $Q \subset P_\omega \otimes P_\omega^{op}$ generated by $x_i \otimes x_i$. It is easy to see that the algebra Q is *commutative*. Thus, setting $x_i \otimes x_i$ to be equal to any numbers $\beta_i \in \mathbb{C}$, we can obtain commuting difference operators $\tilde{\mathcal{M}}_1, ..., \tilde{\mathcal{M}}_r$ with *scalar* coefficients. We will fix the normalization by letting $\beta_i = -1$, and define an system of r commuting difference operators $\mathcal{M}_i := e^{(\rho,h)}\tilde{\mathcal{M}}_i e^{-(\rho,h)}$, $i = 1, ..., r$,

The fact that these difference operators are algebraically independent follows from the facts that any central element of $U(\mathfrak{g})$ can be q-deformed, and that the differential operators of Section 1 are algebraically independent.

We will call the system $\{\mathcal{M}_i\}$ *the q-deformed Toda system*. We note that this system depends on the choice of the orientation ω of the Dynkin diagram.

4. q-deformed affine Toda systems

In this section we will generalize the constructions of Section 3 to quantum affine algebras.

Consider the quantum affine algebra $U_\hbar(\hat{\mathfrak{g}})$. It has generators e_i, f_i, h_i, $i = 0, ..., r$, and the standard relations [CP]. Let $U_\hbar(\hat{\mathfrak{n}}_+)$ be the subalgebra of $U_\hbar(\hat{\mathfrak{g}})$ generated by e_i, and $U_\hbar(\hat{\mathfrak{n}}_-)$ be the subalgebra generated by f_i, respectively.

Let $A_{U_\hbar(\hat{\mathfrak{g}})} = U_\hbar(\hat{\mathfrak{g}})^*$ be the space of linear functions on $U_\hbar(\hat{\mathfrak{g}})$, i.e. the dual quantum formal group.

The obvious two-sided action of $U_\hbar(\hat{\mathfrak{g}})$ on itself induces a two-sided action of $U_\hbar(\hat{\mathfrak{g}})$ on $A_{U_\hbar(\hat{\mathfrak{g}})}$, via $((a,b)\circ\phi)(x) = \phi(axb)$ (here the action of a is a right action and the action of b is a left action). Define the maps $L, R : U_\hbar(\hat{\mathfrak{g}}) \to \mathrm{End}(A_{U_\hbar(\hat{\mathfrak{g}})})$ by $(a,b)\circ\phi = R(a)L(b)\phi$.

For any complex number k, let $A^k_{U_\hbar(\hat{\mathfrak{g}})} \subset A_{U_\hbar(\hat{\mathfrak{g}})}$ be the space of functions satisfying the equation $L(c)f = kf$. Then it is clear that for any $Y \in U_\hbar(\hat{\mathfrak{g}})$, $L(Y)$ preserves $A^k_{U_\hbar(\hat{\mathfrak{g}})}$, and $L(c)|_{A^k_{U_\hbar(\hat{\mathfrak{g}})}} = k \cdot Id$. Thus L descends to a map $L : U_\hbar(\hat{\mathfrak{g}})/(c-k) \to \mathrm{End}_\mathbb{C}(A^k_{U_\hbar(\hat{\mathfrak{g}})})$.

As in the classical case, the value $k = -h^\vee$ is especially important, since at this point there are a lot of interesting central elements. More specifically, one can generalize the Drinfeld-Reshetikhin construction of central elements (see Section 3) to the affine case. This was originally done in [RS] (see also [DE]). In this generalization, to every finite dimensional representation V of $U_\hbar(\hat{\mathfrak{g}})$ one assigns a central element $C_V \in \hat{U}_\hbar(\hat{\mathfrak{g}})/(c+h^\vee)$, where $\hat{U}_\hbar(\hat{\mathfrak{g}})$ is the completion of the quantum affine algebra acting in modules which are locally nilpotent under e_i. This defines a homomorphism from the Grothendieck algebra of the category of finite dimensional

representations of $U_\hbar(\hat{\mathfrak{g}})$ to $\hat{U}_\hbar(\hat{\mathfrak{g}})$. The element C_V is defined by the formula

$$C_V = Res_{z=0} Tr_V (Id \otimes \pi_{V(z)})(\mathcal{R}^{21}\mathcal{R}(1 \otimes e^{2h\rho}))\frac{dz}{z}, \tag{4.1}$$

where $\mathcal{R}$ is the truncated R-matrix of the quantum affine algebra (defined by formula (1.1) in [DE]), and $V(z)$ is the representation V shifted by $z \in \mathbb{C}^*$ (see [DE]).

Now we would like to define the notion of a Whittaker function. In order to do so, we will define an analogue of the notion of a nondegenerate character, like in Section 3.

Let (a_{ij}) $(i, j \geq 0)$ be the Cartan matrix of $\hat{\mathfrak{g}}$. Let $d_i = 1, 2, 3$ be as in section 3 for $i > 0$, and $d_0 = 1$. Let $b_{ij} = d_i a_{ij}$. Let ω be an orientation of the Dynkin diagram of $\hat{\mathfrak{g}}$. Define the quantum polynomial algebra P_ω generated by variables $x_0, x_1, ..., x_r$ corresponding to vertices of the Dynkin diagram, with relations $x_i x_j = e^{\pm\hbar b_{ij}} x_j x_i$, where the sign is $+$ if the edge ij is oriented from i to j and $-$ otherwise.

We will need the following proposition, which is the affine analog of Proposition 3.1.

Proposition 4.1. *There exists a homomorphism $\chi_+ : U_\hbar(\hat{\mathfrak{n}}_+) \to P_\omega$ such that $\chi_+(e_i) = x_i$.*

Proof. Same as Proposition 3.1. □

We also let $\chi_- : U_\hbar(\hat{\mathfrak{n}}_-) \to P_\omega$ be the homomorphism defined by $\chi_-(f_i) = x_i$.

Introduce the algebra $A_\omega = P_\omega \otimes A^{-h^\vee}_{U_\hbar(\hat{\mathfrak{g}})}$.

Definition. *A Whittaker function on $U_\hbar(\hat{\mathfrak{g}})$ is an element $\phi \in A_\omega$ such that for any $a_\pm \in U_\hbar(\hat{\mathfrak{n}}_\pm)$ one has*

$$(a_-, a_+) \circ \phi = \chi_+(a_+)\phi\chi_-(a_-). \tag{4.2}$$

As before, the space of Whittaker functions is naturally identified with $P_\omega \otimes U_\hbar(\mathfrak{h})$. Namely, any element of $P_\omega \otimes U_\hbar(\mathfrak{h})$ can be uniquely extended by equivariance to a Whittaker function. We denote this identification by $\phi \to \phi|_\mathfrak{h}$.

Now consider the fundamental representations $V_1, ..., V_r$ of $U_\hbar(\hat{\mathfrak{g}})$ (see [CP]; they can be bigger than the fundamental representations of $U_\hbar(\mathfrak{g})$ if $\mathfrak{g}$ is not of type A), and define $Y_i = C_{V_i}$.

Let ω be an acyclic orrientation of the Dynkin diagram of $\hat{\mathfrak{g}}$ ("acyclic" is vacuous unless $\mathfrak{g}$ is of type A).

Proposition 4.2. *(i) For any $i = 1, ..., r$ the series $(L(Y_i)\phi)|_\mathfrak{h}$ is finite for any Whittaker function ϕ. Moreover, there exists a unique difference operator $D_{Y_i} : A_\mathfrak{h} \to A_\mathfrak{h}$ on H such that for any Whittaker function ϕ one has $(L(Y_i)\phi)|_\mathfrak{h} = D_{Y_i}\phi|_\mathfrak{h}$.*

(ii) $[D_{Y_i}, D_{Y_j}] = 0$.

Proof. The finiteness of the series is obtained like in Section 2, using a suitable analog of the fact that terms containing non-simple roots vanish (see Section 5 for a detailed proof). The rest of the proof is analogous to the proof of Proposition 1.3. □

Thus D_{Y_i} are B-valued commuting difference operators, where $B = P_\omega \otimes P_\omega^{op}$.

As before, since C_V has zero weight for any V, the element D_Y for $Y = C_V$ is in fact a difference operator with coefficients in the subalgebra $Q \subset P_\omega \otimes P_\omega^{op}$ generated by $x_i \otimes x_i$. The algebra Q is commutative. Thus, setting $x_i \otimes x_i$ to be equal to any numbers $\beta_i \in \mathbb{C}$, we can obtain commuting difference operators $\tilde{\mathcal{M}}_1^K, ..., \tilde{\mathcal{M}}_r^K$ with scalar coefficients. We will fix the normalization by letting $\beta_i = -1$, $i \geq 1$, $\beta_0 = -K$, and define a system of r commuting difference operators $\mathcal{M}_i^K = e^{(\rho,h)}\tilde{\mathcal{M}}_i^K e^{-(\rho,h)}$, $i = 1, .., r$.

As before, the operators $\mathcal{M}_i^K$ are algebraically independent. This follows from the fact that by Section 3 this is the case for $K = 0$.

We will call the system $\{\mathcal{M}_i^K\}$ *the q-deformed affine Toda system.* It depends on the orientation on the Dynkin diagram.

5. Computation of the q-deformed Toda operators

In this section we will complete the proof of Proposition 4.2, explain how to compute the q-deformed and q-deformed affine Toda operators, and compute some of them for $\mathfrak{g} = sl(N)$.

First of all, we need to recall the notion of Cartan-Weyl root elements e_β of a finite dimensional or affine quantum group (corresponding to all roots), due to Khoroshkin and Tolstoy [KhT1,KhT2]. To define them, one needs to fix a normal ordering of positive roots (see [T]). (As was shown in [T], such an ordering can be obtained by extending any ordering of simple roots). Then one computes the corresponding elements e_β by formulas (10-14) of [KhT1].

Propsoition 5.1. *Suppose that an orientation ω of the Dynkin diagram is consistent with the ordering of simple roots used to define the Cartan-Weyl root vectors (i.e. edges are oriented from smaller to larger simple roots). Then the homomorphisms $\chi_\pm$ corresponding to ω annihilate the Cartan-Weyl root vectors for non-simple roots.*

Proof. This follows from the definition of the Cartan-Weyl generators, i.e. formulas (10-14) in [KhT1]. (The Cartan-Weyl generators are defined as iterated q-commutators of simple root elements, so they by definition map to 0 under $\chi_\pm$). □

Next, we prove the following lemma about Whittaker functions.

We consider the quantum group $U_\hbar(\mathfrak{g})$ or $U_\hbar(\hat{\mathfrak{g}})$. Let us fix an acyclic orientation ω of its Dynkin diagram and extend it to an ordering of simple positive roots. Let us further extend this ordering to a normal ordering of positive roots.

Lemma 5.2. *Let $X = f_{\gamma_1}...f_{\gamma_n}e_{\gamma'_1}...e_{\gamma'_m}$, where f_γ, e_γ are root elements from the Cartan-Weyl basis corresponding to roots $-\gamma$ and γ ($\gamma > 0$). If the roots $\gamma_1, ..., \gamma_n, \gamma'_1, ..., \gamma'_m$ are not all simple then $L(X)\phi = 0$ for any Whittaker function ϕ (where to define Whittaker functions one used the orientation ω). Otherwise, if $\gamma_i = \alpha_{k_i}$, $\gamma'_i = \alpha_{l_i}$ then*

$$(L(X)\phi)|_{\mathfrak{h}} = e^{(\sum \alpha_{k_i},h)}x_{l_1}...x_{l_m}\phi|_{\mathfrak{h}}x_{k_n}...x_{k_1} \tag{5.1}$$

Proof. By Proposition 5.1 we have: $L(e_\alpha)\phi$ equals 0 if α is not simple and $x_i\phi$ if $\alpha = \alpha_i$. Therefore, it is easy to see that in order for $L(X)\phi$ to be nonzero, the roots γ'_i must be simple. If they are (i.e. $\gamma'_i = \alpha_{l_i}$) then we have

$$(L(X)\phi)|_{\mathfrak{h}} = x_{l_1}...x_{l_m}(L(f_{\gamma_1}...f_{\gamma_n})\phi)|_{\mathfrak{h}}. \tag{5.2}$$

Now, we have (like in the proof of Proposition 1.3):

$$(L(f_{\gamma_1}...f_{\gamma_n})\phi)|_{\mathfrak{h}} = e^{(\sum \gamma_i, h)}(R(f_{\gamma_1}...f_{\gamma_n})\phi)|_{\mathfrak{h}}.$$

Thus, similarly to the above, we get 0 if at least one γ_i is not simple. If $\gamma_i = \alpha_{k_i}$ then we get (5.1). $\square$

Now we are ready to compute the q-deformed Toda operators. The computation is based on the explicit formula for the R-matrix $\mathcal{R}$ given in [KhT1] (formula (42)). This formula says that $\mathcal{R}$ (for finite dimensional or affine quantum groups) can be represented as a normally ordered product of factors corresponding to positive roots (for any fixed normal ordering).

So let us fix a normal ordering as above and consider the Khoroshkin-Tolstoy representation of $\mathcal{R}$. Let $\mathcal{R}_s$ be the defined by the same product as $\mathcal{R}$ but with all terms corresponding to non-simple roots crossed out. That is,

$$\mathcal{R}_s = (\prod_i \mathcal{R}_{\alpha_i})\mathcal{K}, \tag{5.3}$$

where $\mathcal{K} = q^{\sum_{i=1}^r y_i \otimes y_i}$, for any orthonormal basis y_i of $\mathfrak{h}$, and $\mathcal{R}_{\alpha_i}$ are R-matrices corresponding to simple roots:

$$\mathcal{R}_{\alpha_i} = \exp_{q^{2d_i}}((q^{d_i} - q^{-d_i})e_{\alpha_i} \otimes f_{\alpha_i}), \tag{5.4}$$

where

$$\exp_p(x) = \sum_{n\geq 0} \frac{(p-1)^n}{(p-1)...(p^n-1)} x^n \tag{5.5}$$

is the quantum exponential.

Remark. The product in (5.3) is ordered according to the ordering of simple roots obtained by extension of ω. It is not hard to show, however, that it depends only on ω itself and not on the extension (This follows from the obvious fact that any two complete orders extending a partial order on a finite set can be identified by a sequence of transpositions of neighbors where transposed elements are not comparable).

We have the following obvious corollary from Lemma 5.2.

Corollary 5.3. *Let C_V^s be defined by the same formula as C_V (i.e. (3.1) or (4.1)) but with $\mathcal{R}$ replaced by $\mathcal{R}_s$. Let $Y_i^s = C_{V_i}^s$. Then $D_{Y_i^s} = D_{Y_i}$.*

The operators $D_{Y_i^s}$ are relatively easy to compute in any given special case, since they contain contributions from simple roots only. In the remainder of this section, we will restrict ourselves to $\mathfrak{g} = sl(N)$ and let $V_i = \Lambda^i V$, where V is the vector representation. We will compute the q-Toda operator corresponding to $i = 1$ (i.e. $V_i = V$) explicitly.

First of all, we will consider the non-affine case. We have $e_i^2 = f_i^2 = 0$ in V, and so in the definition of $D_{Y_1^s}$ one can replace $\mathcal{R}_{\alpha_i}$ with the first two terms $1+X_i$, where $X_i = (q-q^{-1})e_i \otimes f_i$. Thus, $D_{Y_1^s}$ is obtained from a product of such binomial terms. After opening the brackets, we find that many terms are equal to zero. Indeed, we have

$$C_V = \sum_l \sum_{i_1,...,i_l} \mathrm{Tr}|_V (X_{i_1}^{21}....X_{i_l}^{21} X_{i_1}...X_{i_l} q^{2\rho})$$

It is clear that a term in this sum can be nonzero only if $i_1, ..., i_l$ are consequtive (as it contains $e_{i_1}...e_{i_l}|_V$) and $i_l, ..., i_1$ are consequtive (as it contains $f_{i_1}...f_{i_l}|_V$). Thus, the terms with $l > 1$ are zero. Computing the terms with $l = 0, 1$, we obtain

$$C_V = \sum_{j=1}^{N} q^{N+1-2j} q^{2\omega_j} + (q - q^{-1})^2 \sum_{i=1}^{N-1} q^{N+1-2i} f_i q^{\omega_{i+1}} e_i q^{\omega_i}, \tag{5.6}$$

where ω_i is the weight of the i-th basis vector of V. From (5.6), we get

$$\mathcal{M}_1 = \sum_{j=1}^{N} T_j^2 - (q - q^{-1})^2 \sum_{i=1}^{N-1} e^{(h,\alpha_i)} T_i T_{i+1}, \tag{5.7}$$

where $T_i = T_{\omega_i}$.

Now consider the affine case. In this case, using a similar argument to the above, instead of (5.6) we get

$$C_V = \sum_{j=1}^{N} q^{N+1-2j} q^{2\omega_j} + (q - q^{-1})^2 \sum_{i=1}^{N} q^{N+1-2i} f_i q^{\omega_{i+1}} e_i q^{\omega_i}, \tag{5.8}$$

with subscripts understood modulo N, i.e. $0 = N$. (here we use that, according to (4.1), we only take the zero degree terms in the z-expansion of $(1 \otimes \pi_{V(z)})(\mathcal{R}^{21}\mathcal{R})$). Therefore, instead of (5.8) we get

$$\mathcal{M}_1^K = \sum_{j=1}^{N} T_j^2 - (q - q^{-1})^2 \sum_{i=1}^{N} K^{\delta_{iN}} e^{(h,\alpha_i)} T_i T_{i+1}. \tag{5.9}$$

Similarly one can compute $\mathcal{M}_i$, $\mathcal{M}_i^K$, $i > 1$ with a somewhat more complicated answer.

6. Quasiclassical limit of q-deformed quantum Toda systems, and their relation to quantum relativistic Toda systems

The following proposition explains why we refer to the quantum integrable systems of Sections 3,4 as q-deformed (non-affine or affine) Toda systems.

Proposition 6.1. *For any i and K one has*

$$\lim_{\hbar \to 0} \frac{\mathcal{M}_i^K - dim(V_i)}{(q - q^{-1})^2} = C_i(M^K + G_i), \tag{6.1}$$

where $C_i \in \mathbb{C}^*$ and $G_i \in \mathbb{C}$.

Proof. This proposition follows from Theorem 4.1 of [DE], which computes the quasiclassical limit of central elements. □

Remark. In the special case $\mathfrak{g} = sl(N)$, $i = 1$, Proposition 6.1 can be checked directly from (5.9): it is easy to see that

$$\lim_{\hbar \to 0} \frac{\mathcal{M}_1^K - N}{(q - q^{-1})^2} = -M_1^K, \tag{6.2}$$

Next, let us show that the q-deformed quantum Toda systems of Sections 3,4 (for $\mathfrak{g} = sl(N)$) are equivalent to quantum relativistic Toda systems of [Ru1], nonperiodic and periodic, respectively.

Introduce coordinates $z_1, ..., z_N$ so that $\mathfrak{h}$ is the set of solutions of $\sum z_i = 0$. We will realize formal functions on H as functions of $z_1, ..., z_N$ invariant under simultaneous shift of z_i. In terms of z_i, the operator $\mathcal{M}_1^K$ (as operator on such functions) can be written in the form

$$\mathcal{M}_1^K = \sum_{i=1}^N T_i^2 - (q-q^{-1})^2 \sum_{i=1}^N K^{\delta_{iN}} e^{z_i - z_{i+1}} T_i T_{i+1}, \tag{6.3}$$

where $T_i f(z_1, ..., z_i, ..., z_N) = f(z_1, ..., z_i + \hbar, ..., z_N)$ (we remind that subscripts are understood cyclically).

Consider the algebra Q of operators on simultaneous-shift-invariant functions generated by $e^{\pm z_i \mp z_{i+1}}$ and $T_i^{\pm 1}$. It is clear that for all i the operator $\mathcal{M}_i^K$ belongs to Q. Consider the automorphism of Q defined by $T_i \to T_i$, $e^{z_i - z_{i+1}} \to e^{z_i - z_{i+1}} T_i T_{i+1}^{-1}$. Under this automorphism, the operator $\mathcal{M}_1^K$ is mapped to a simpler operator

$$\bar{\mathcal{M}}_1^K = \sum_{i=1}^N \mathcal{T}_i - (q-q^{-1})^2 \sum_{i=1}^N K^{\delta_{iN}} e^{z_i - z_{i+1}} \mathcal{T}_i, \tag{6.4}$$

where $\mathcal{T}_i := T_i^2$.

Thus, the q-deformed affine quantum Toda system for $sl(N)$ is equivalent (as an integrable system) to the system defined by the Hamiltonian (6.4). Setting $K = 0$, we get that the q-deformed non-affine quantum Toda system is equivalent to the system defined by

$$\bar{\mathcal{M}}_1 = \sum_{i=1}^N \mathcal{T}_i - (q-q^{-1})^2 \sum_{i=1}^{N-1} e^{z_i - z_{i+1}} \mathcal{T}_i, \tag{6.5}$$

Now recall from [Ru1] that the Hamiltonian of the quantum relativistic Toda system is

$$\hat{S}_1 = \sum_{i=1}^N f(z_{i-1} - z_i) \mathcal{T}_i f(z_i - z_{i+1}), \tag{6.6}$$

where $f(a) := (1 + g^2 e^a)^{1/2}$, and $z_0 = z_N$, $z_{N+1} = z_1$ in the periodic case, and $z_0 = -\infty$, $z_{N+1} = +\infty$ in the non-periodic case. Let us conjugate $\hat{S}_1$ by the function $\prod_i \psi(z_i - z_{i+1})$, where ψ satisfies the difference equation $\psi(x + 2\hbar) = \psi(x) f(x)^{-1}$, and the product is from 1 to N in the periodic case, and from 1 to $N-1$ in the nonperiodic case. We get

$$\psi^{-1} \hat{S}_1 \psi := \bar{S}_1 = \sum_{i=1}^N (1 + g^2 e^{z_i - z_{i+1}}) T_i. \tag{6.7}$$

In the nonperiodic case, setting $g = \sqrt{-1}(q - q^{-1})$, we see that (6.7) becomes (6.5). In the periodic case, we set $g = \sqrt{-1}(q-q^{-1})K^{1/N}$, and $z_i' = z_i - \frac{i}{N} \ln K$, where

$K \neq 0$ is arbitrary. Then (6.7) becomes (6.4). This demonstrates the equivalence of the q-deformed and relativistic Toda lattices.

Remark 1. In the non-periodic relativistic Toda lattice, the parameter g can be removed by a shift of variables, while in the periodic case it is an essential parameter. This corresponds to the presence of K in the the affine and its absence in the non-affine case.

Remark 2. Note that our quantum group theoretic procedure of Sections 3 and 4 yields the Hamiltonian given by (6.3) and not the simpler Hamiltonian (6.4). In fact, one can get (6.4) instead of (6.3) by a slight modification of the procedure.

Namely, if H_i are any elements of $\mathfrak{h}$ such that $\alpha_i(H_j) = \alpha_j(H_i)$ then new elements $e_i' = e_i e^{\hbar H_i}$ satisfy the quantum Serre relations and commutativity for orthogonal roots. Instead of the algebra $U_\hbar(\hat{\mathfrak{n}}_+)$ used in our argument, we could use $U_q(\hat{\mathfrak{n}}_+')$, generated by e_i', and similarly for $U_\hbar(\hat{\mathfrak{n}}_-)$. This would produce a different operator from (6.3), which is, however, equivalent to (6.3) by an automorphism of the algebra of difference operators. In particular, it is easy to see that one can choose H_i so that the obtained operator is (6.4).

In this connection we would like to mention the paper [S], where a similar idea is used: the author finds elements H_i such that the elements e_i' generate an algebra which has nondegenerate characters (such H_i are determined from a system of linear nonhomogeneous equations, whose homogeneous part is the above system $\alpha_i(H_j) = \alpha_j(H_i)$). Using such characters, one may also define q-deformed quantum Toda systems. This approach is closely related to ours. For example, in [S] the system for H_i depends on a choice of a Coxeter element, which in our situation corresponds to the necessity to choose an orientation of the Dynkin diagram.

7. Toda systems as limits of Calogero-Moser, Macdonald, and Ruijsenaars systems.

In this section we discuss the limiting procedures which allow to obtain quantum Toda systems as limiting cases of more complicated integrable systems inbvolving all roots of a Lie algebra rather than just simple roots. The results presented here are known and are given only for the sake of completeness of the picture.

It is known [I] that quantum Toda systems can be represented as limits of quantum Calogero-Moser systems.

Namely, for any finite dimensional Lie algebra $\mathfrak{g}$ consider the quantum trigonometric Calogero-Moser Hamiltonian

$$H_T(k) = -\frac{1}{2}\Delta + k(k-1)\sum_{\alpha>0}\frac{1}{\sinh^2\alpha(h)}. \tag{7.1}$$

Set $k = \frac{1}{2}e^P$, $h = -\frac{1}{2}x + P\rho$. In terms of the new notation, we have

$$H_T(k) = -\frac{1}{2}\Delta + e^P(e^P-2)\sum_{\alpha>0}\frac{1}{4\sinh^2(-\frac{1}{2}\alpha(x) + P(\alpha,\rho))}, \tag{7.2}$$

This implies that

$$\lim_{P\to+\infty} H_T(k) = -\frac{1}{2}\Delta + \sum_{i=1}^{r} e^{\alpha_i(h)} = M. \tag{7.3}$$

(Indeed, only terms corresponding to simple roots remain finite: the term corresponding to a root $\alpha > 0$ behaves like const$\cdot e^{2P(1-(\alpha,\rho))}$). This proves our statement in the non-affine case.

For the affine case, we consider the elliptic Calogero-Moser Hamiltonian

$$H_E(k) = -\frac{1}{2}\Delta - 4\pi^2 k(k-1)\sum_{\alpha>0}\wp(\frac{\alpha(h)}{2\pi i}, it), \tag{7.4}$$

where $\wp$ is the Weierstrass elliptic function with periods $1, it$, and $t > 0$. Set $k = \frac{1}{2}e^P$, and $h = -\frac{1}{2}x + P\rho$, $t = Ph^\vee - \frac{1}{2}\ln K$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Recall that

$$-4\pi^2\wp(\frac{z}{2\pi i}, it) = \sum_{n\in\mathbb{Z}}\sinh^{-2}(z+tn) + \gamma(t). \tag{7.5}$$

Let m be the number of positive roots of $\mathfrak{g}$.

We have

$$\begin{aligned} &H_E(k) - mk(k-1)\gamma(t) = \\ &-\frac{1}{2}\Delta + \frac{1}{4}e^P(e^P-2)\sum_{\alpha>0, n\in\mathbb{Z}}\sinh^{-2}(-\frac{1}{2}\alpha(x) + P(\alpha,\rho) + nPh^\vee - \frac{n}{2}\ln K) \end{aligned} \tag{7.6}$$

(for simplicity we assume that $K > 0$).

Therefore,

$$\lim_{P\to+\infty}(H_E(k) - mk(k-1)\gamma(t)) = -\frac{1}{2}\Delta + \sum_{i=0}^{r} K^{\delta_{i0}} e^{\alpha_i(h)} = M^K. \tag{7.7}$$

(Now the surviving roots are not only the simple roots of $\mathfrak{g}$ but also the maximal root, because by the definition $h^\vee = 1 + \theta(\rho)$). This proves our claim in the affine case.

This limiting procedure can be used to give another proof of Theorem 2.1 which does not use quantum groups. Namely, it was shown by Cherednik [Ch1] that the quantum system defined by the Hamiltonian (7.4) is integrable for all Lie algebras. By a limiting argument, one can deduce from this that the same is true for the Hamiltonian M^K.

Similarly, it is known that q-deformed quantum Toda systems for $\mathfrak{g} = sl(N)$ can be viewed as limits of quantum Macdonald-Ruijsenaars systems defined in [Mac,Ru2].

Consider the non-affine case. Recall [Mac,Ru2] that the trigonometric Macdonald-Ruijsenaars system is defined by the quantum Hamiltonian

$$H_T^q = \sum_{i=1}^{N}(\prod_{j\neq i}\frac{q^{2k}e^{w_i} - e^{w_j}}{e^{w_i} - e^{w_j}})\mathcal{T}_i. \tag{7.8}$$

Let us conjugate H_T^q with the function $\eta = e^{-P\sum_{i=1}^N (i-1)w_i}$. We get

$$\hat{H}_T^q = \eta^{-1}H_T^q\eta = \sum_{i=1}^{N} q^{-2(i-1)P}(\prod_{j\neq i}\frac{q^{2k}e^{w_i} - e^{w_j}}{e^{w_i} - e^{w_j}})\mathcal{T}_i. \tag{7.9}$$

Set $w_i = z_i + 2\hbar i P$, $k = P$. We get

$$\hat{H}_T^q = \sum_{i=1}^N e^{-2\hbar(i-1)P}(\prod_{j\neq i} \frac{e^{2\hbar(i+1)P}e^{z_i} - e^{2\hbar jP}e^{z_j}}{e^{2\hbar iP}e^{z_i} - e^{2\hbar jP}e^{z_j}})\mathcal{T}_i. \tag{7.10}$$

Now let $\hbar P \to +\infty$ (here $\hbar$ has to be a complex number, not a formal parameter). It is easy to see that

$$\hat{H}_T^q \to \mathcal{T}_N + \sum_{i=1}^{N-1}(1 - e^{z_i - z_{i+1}})\mathcal{T}_i. \tag{7.11}$$

This coincides with (6.5) after a shift of variables z_i.

A similar computation shows that the Hamiltonian given by (6.4) can be obtained by a limiting procedure from the Ruijsenaars' relativistic elliptic Calogero-Moser Hamiltonian [Ru2].

Remark 1. We expect that similar results are the case for all Lie algebras $\mathfrak{g}$. Namely, we expect that the non-affine and affine q-Toda systems can be obtained as a limit of trigonometric, respectively elliptic Macdonald-Ruijsenaars operators, which were defined for an arbitrary root system by Cherednik [Ch2,Ch3].

Remark 2. In [E], it is shown that Calogero-Moser operators (trigonometric and elliptic) are obtained as radial parts of central elements of $U(\mathfrak{g})$ (respectively, of a completion of $U(\hat{\mathfrak{g}})$ at the critical level) on equivariant functions on G (respectively, $\hat{G}$) with values in certain special representations U_k. In [EK], it is shown that Macdonald operators are obtained in a similar manner as radial parts of central elements of $U_\hbar(\mathfrak{g})$. In view of these results and the results of this paper, it would be tempting to understand the above limiting procedures in terms of representation theory (i.e. to see how equivariant functions with values in U_k turn into Whittaker functions in the limit).

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